MAT3027
PROBABILITY AND RANDOM VARIABLES

INTRODUCTION

WEEK # 1

COURSE DETAILS

- Instructor: M. ASLI AYDIN
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  - Office: D405
  - Office Hours: By appointment (at least two days in advance)
- Lectures:
  - Tuesday 08:30-11:20 @A102
  - Tuesday 15:30-18:20 @B402

About Course

- Learning foundations of probability theory
- Learning modelling and analysis of random events encountered in (industrial) engineering applications

Prerequisites

There is NO formal prerequisite BUT you need to know

- CALCULUS: Derivative, Integral, Double Integral of polynomials, trigonometric fn.s, Exponential fn.s, etc.

Topics

Week	Subject
1)	Combinatorial Analysis
2)	Axioms of Probability
3)	Conditional Probability and Independence, Bayes Theorem
4)	The Concept of Random Variables (RVs), Expectation, Variance
5)	Discrete RVs
6)	Continuous RVs
7)	Special Discrete RVs
8)	Special Continuous RVs
9)	Midterm
10)	Functions of RVs
11)	Jointly Distributed RVs
12)	Sums of RVs
13)	Sums of RVs
14)	Limit Theorems

Recommended Books

- Ross, Sheldon. "A First Course in Probability", Pearson
- Walpole, R.E. et al. , "Probability and Statistics for Engineers and Scientists", Pearson

Grading

- (Random Online) Quizzes (20%)

- Midterm (35%)

  At week #8 or #9, from topics covered so far.

- Final (45%)

  From topics covered throught the semester

Make-Up Policy

Midterm & Final

- Contact with the Dean's Secretary with a valid report of your excuse.

Quizzes

- No Make-Up ☹

Attendance

- I will not take attendance so there is not NA Grades.

BUT

I highly recommend you to attend/follow the lectures to understand the topics better.

Lecture slides may NOT cover all the information you need to learn.

Privacy and Copy Rights

- In accordance with the Personal Data Protection Law,

it is forbidden to **take the photos** and **record the videos** of the

participants (both students and instructors) during the lectures.

Combinatorial Analysis

The basic principle of counting

Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

Example 2a A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are $10 \times 3 = 30$ possible choices. ■

The generalized basic principle of counting

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if …, then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

Example 2c

How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Solution By the generalized version of the basic principle, the answer is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175{,}760{,}000$. ■

Example 2d

How many functions defined on n points are possible if each functional value is either 0 or 1?

Solution Let the points be $1, 2, \ldots, n$. Since $f(i)$ must be either 0 or 1 for each $i = 1, 2, \ldots, n$, it follows that there are 2^n possible functions. ■

Permutations

How many different ordered arrangements of the letters a, b, and c are possible? By direct enumeration we see that there are 6, namely, *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*. Each arrangement is known as a *permutation*. Thus, there are 6 possible permutations of a set of 3 objects. This result could also have been obtained from the basic principle, since the first object in the permutation can be any of the 3, the second object in the permutation can then be chosen from any of the remaining 2, and the third object in the permutation is then the remaining 1. Thus, there are $3 \cdot 2 \cdot 1 = 6$ possible permutations.

Permutations

Suppose now that we have n objects. Reasoning similar to that we have just used for the 3 letters then shows that there are

$$n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1 = n!$$

different permutations of the n objects.

Permutations

Example 3c

Ms. Jones has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

Solution There are $4!\ 3!\ 2!\ 1!$ arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each possible ordering of the subjects, there are $4!\ 3!\ 2!\ 1!$ possible arrangements. Hence, as there are $4!$ possible orderings of the subjects, the desired answer is $4!\ 4!\ 3!\ 2!\ 1! = 6912$. ■

Permutations

Example 3d How many different letter arrangements can be formed from the letters *PEPPER*?

Solution We first note that there are 6! permutations of the letters $P_1E_1P_2P_3E_2R$ when the 3P's and the 2E's are distinguished from one another. However, consider any one of these permutations—for instance, $P_1P_2E_1P_3E_2R$. If we now permute the P's among themselves and the E's among themselves, then the resultant arrangement would still be of the form *PPEPER*. That is, all 3! 2! permutations

$$\begin{array}{ll} P_1P_2E_1P_3E_2R & P_1P_2E_2P_3E_1R \\ P_1P_3E_1P_2E_2R & P_1P_3E_2P_2E_1R \\ P_2P_1E_1P_3E_2R & P_2P_1E_2P_3E_1R \\ P_2P_3E_1P_1E_2R & P_2P_3E_2P_1E_1R \\ P_3P_1E_1P_2E_2R & P_3P_1E_2P_2E_1R \\ P_3P_2E_1P_1E_2R & P_3P_2E_2P_1E_1R \end{array}$$

are of the form *PPEPER*. Hence, there are $6!/(3!\,2!) = 60$ possible letter arrangements of the letters *PEPPER*. ■

Permutations

In general, the same reasoning as that used in Example 3d shows that there are

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, $\ldots, n_r$ are alike.

Combinations

We are often interested in determining the number of different groups of r objects that could be formed from a total of n objects.

Notation and terminology

We define $\binom{n}{r}$, for $r \leq n$, by

$$\binom{n}{r} = \frac{n!}{(n - r)!\, r!}$$

and say that $\binom{n}{r}$ (read as "n choose r") represents the number of possible combinations of n objects taken r at a time.†

Combinations

Example 4a A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Solution There are $\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140$ possible committees. ■

Combinations

$\wedge 1 \wedge 1 \wedge 1 \ldots \wedge 1 \wedge 1 \wedge$

1 = functional

$\wedge$ = place for at most one defective

Example 4c Consider a set of n antennas of which m are defective and $n - m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

Solution Imagine that the $n - m$ functional antennas are lined up among themselves. Now, if no two defectives are to be consecutive, then the spaces between the functional antennas must each contain at most one defective antenna. That is, in the $n - m + 1$ possible positions—represented in Figure 1.1 by carets—between the $n - m$ functional antennas, we must select m of these in which to put the defective antennas. Hence, there are $\binom{n-m+1}{m}$ possible orderings in which there is at least one functional antenna between any two defective ones. ■

A president and a treasurer are to be chosen from a student club consisting of 50 people. How many different choices of officers are possible if

(a) there are no restrictions;

(b) A will serve only if he is president;

(c) B and C will serve together or not at all;

(d) D and E will not serve together?

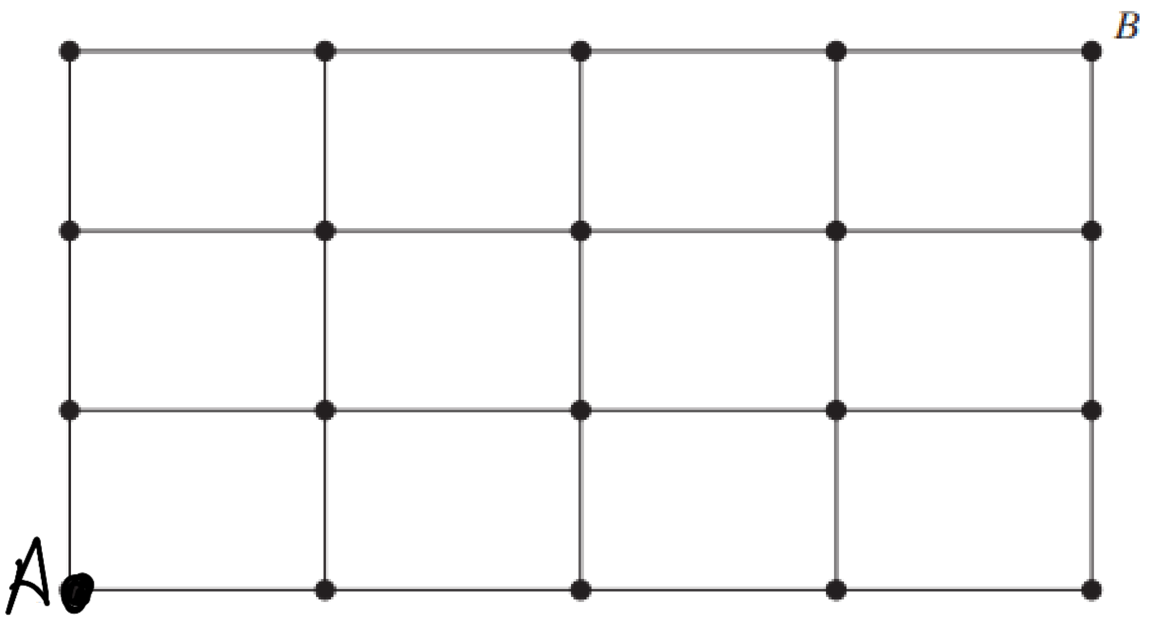
(a) The total number of choices of officers, without any restrictions, is

$$_{50}P_2 = \frac{50!}{48!} = (50)(49) = 2450.$$

(b) Since A will serve only if he is president, we have two situations here: (i) A is selected as the president, which yields 49 possible outcomes for the treasurer's position, or (ii) officers are selected from the remaining 49 people without A, which has the number of choices $_{49}P_2 = (49)(48) = 2352$. Therefore, the total number of choices is $49 + 2352 = 2401$.

(c) The number of selections when B and C serve together is 2. The number of selections when both B and C are not chosen is $_{48}P_2 = 2256$. Therefore, the total number of choices in this situation is $2 + 2256 = 2258$.

(d) The number of selections when D serves as an officer but not E is $(2)(48) = 96$, where 2 is the number of positions D can take and 48 is the number of selections of the other officer from the remaining people in the club except E. The number of selections when E serves as an officer but not D is also $(2)(48) = 96$. The number of selections when both D and E are not chosen is $_{48}P_2 = 2256$. Therefore, the total number of choices is $(2)(96) + 2256 = 2448$. This problem also has another short solution: Since D and E can only serve together in 2 ways, the answer is $2450 - 2 = 2448$. ∎



21. Consider the grid of points shown at the top of the next column. Suppose that, starting at the point labeled A, you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B is reached. How many different paths from A to B are possible?

Hint: Note that to reach B from A, you must take 4 steps to the right and 3 steps upward.

Solution:

Note that the number of paths is equal to the number of ordered 7-tuples with 3 U's and 4 R's, for example $URRURRU$ means, in order, go up, right, right, up, right, right, up.

The number of these is

$$\frac{\#\text{ways to order 7 different letters}}{\#\text{ways to re-order the equivalent } U\text{'s} \cdot \#\text{ways to re-order the equivalent } R\text{'s}} = \frac{7!}{3! \cdot 4!}$$

$$= \binom{7}{4}.$$

In how many ways can 7 graduate students be assigned to 1 triple and 2 double hotel rooms during a conference?

Solution:

The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!},$$

where $n_1 + n_2 + \cdots + n_r = n$.

The total number of possible partitions would be

$$\binom{7}{3, 2, 2} = \frac{7!}{3!\ 2!\ 2!} = 210.$$

References

- Ross, S. "A First Course in Probability", Pearson- CH#1