MAT1051 Midterm Exam
[Group B]
November 25, 2023

Name: ______________________________ **ID:** ______________________

Section: _______________ **Signature:** __________________

- You will find **7 multiple-choice questions** and **3 open-ended questions** on the following pages.
- **Important Information about the multiple choice questions:**
  - Use erasable pencils. For each question there is exactly one correct answer. Mark your answers into the optical “Answer Sheet”. If you do NOT mark your answer into the “Answer Sheet” it will NOT be graded!
  - Be sure that you fill in your information in the “Answer Sheet” correctly, otherwise your exam paper may NOT be graded.
  - **There are two different “Question Booklets” (Group A and Group B). Do NOT forget to mark the “exam paper group” in your “Answer Sheet”.**
  - Below each multiple-choice question, there is enough space for you to carry all calculations before marking your answer in the optical "Answer Sheet."
  - For each wrong answer in the the multiple-choice questions, **1 point will be deducted**.
- **Important Information about the open-ended questions:**
  - For the open-ended questions, write your solutions clearly in the exam paper.
  - Show all your work in all open-ended questions. **No** credit will be given for correct final answers without justification.
- Calculators are **not** allowed.
- Exam duration: 90 minutes.

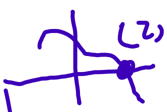
Problem	**Score**	**Points**
MCQ		**35**
Q8		**10**
Q9		**15**
Q10		**10**
Total		**70**

1. (5 points) Only **one** of the following statements is **true** about the function $f(x) = \dfrac{3x+|x|}{x}$.

   A. The function intersects the x-axis at $x=2$.
   B. $f(x)$ is an even function.
   C. The function is differentiable at $x=0$.
   D. The range of the function is $\{2,4\}$.
   E. There is exactly one y-intercept for the function.

مراجعة شيت الاسئلة -

20, 30 [illegible]

$\boxed{A}$  $(2,0)$ $\boxed{0 \neq \dfrac{3.2+|2|}{2}} \Rightarrow 0 \neq 4$ A غلط

$\boxed{B}$ $f(-x) = f(x)$?? $f(-x) = \dfrac{-3x+|-x|}{-x} = \dfrac{-3x+x}{-3} \neq f(x)$. B غلط

$\boxed{C}$ $f(0) = \dfrac{0}{0}$ [illegible]. $f(0)$ undefined $\Rightarrow x=0$ not contains $\Rightarrow x=0$ not defined

$\boxed{E}$ D $f(0)$ [illegible] function zero [illegible] غلط

2. (5 points) Let $f(x) = \dfrac{1}{\sqrt{x}}$ and $g(x) = 4-x^2$. The domain of $(f \circ g)(x)$ is

   A. $(-2,2)$ B. $[-2,2]$ C. $[0,\infty)$ D. $(-\infty,0]$ E. $(-\infty,\infty)$

$f \circ g(x) = f(g(x)) = f(4-x^2) = \dfrac{1}{\sqrt{4-x^2}}$, $\sqrt{4-x^2} \neq 0$, $4-x^2 \geq 0$

Domain $4-x^2 > 0 \Rightarrow (2-x)(2+x) > 0$ → $2+x > 0$ and $2-x > 0$; → $2+x < 0$ and $2-x < 0$

$x > -2$ and $2 > x \Rightarrow (-2,\infty) \cap (-\infty,2) = (-2,2)$

$x < -2$ and $2 < x \Rightarrow (-\infty,-2) \cap (2,\infty) = \emptyset$

domain $= (-2,2) \cup \emptyset = (-2,2)$

3. (5 points) The equation of the tangent line to the curve $y = \sin(x^2)+1$ at the point $\left(\sqrt{\frac{\pi}{2}}, 2\right)$ is

The tangent equation $y - y_1 = m(x-x_1) \Rightarrow y-2 = m\left(x - \sqrt{\frac{\pi}{2}}\right)$

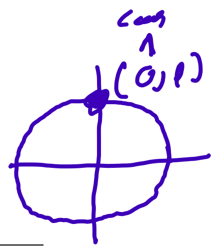
   A. $y = 2\sqrt{\frac{\pi}{2}}x+2$ B. $y = x+2-\sqrt{\frac{\pi}{2}}$ C. $x=2$ D. $y = \pi\left(x-\sqrt{\frac{\pi}{2}}\right)+2$ **E.** $y=2$

m = slope of the tangent at $\left(\sqrt{\frac{\pi}{2}}, 2\right) = y'\Big|_{\left(\sqrt{\frac{\pi}{2}},2\right)}$

$y' = \cos x^2 \cdot 2x = 2x\cos x^2$

$m = y'\Big|_{\left(\sqrt{\frac{\pi}{2}},2\right)} = 2\sqrt{\frac{\pi}{2}}\cos\left(\sqrt{\frac{\pi}{2}}\right)^2 = \sqrt{2}\sqrt{\pi}\cos\frac{\pi}{2} = \sqrt{2}\sqrt{\pi}\cdot 0 = 0$

$\sqrt{2}\sqrt{2}$ $\dfrac{\sqrt{\pi}}{\sqrt{2}}$



The equation of tangent is $y-2 = 0\left(x-\sqrt{\frac{\pi}{2}}\right) \Rightarrow y-2=0 \Rightarrow \boxed{y=2}$

4. (5 points)

$$\lim_{x\to 2} \frac{\sqrt{4-x^2}-\sqrt{x-2}}{\sqrt{x^3+x^2-6x}} =$$

A. $-\sqrt{\frac{3}{10}}$ B. 0 C. $\frac{3}{\sqrt{10}}$ D. ∞ **E.** $\frac{1}{\sqrt{10}}$

5. (5 points) Let

$$f(x) = \begin{cases} 5^{2x-2}+2, & x \le 1 \\ \frac{3x^2+x+5}{Ax^2+2}, & 1 < x < 2 \\ \ln(Bx)+1, & x \ge 2 \end{cases}$$ where A and B are positive constants.

Find all values of A and B such that $f(x)$ is continuous at $x=1$ and $x=2$.

A. $A=1$ and $B=\frac{1}{2}e^2$

B. $A=1$ **and** $B=\frac{1}{2}e^{\frac{13}{6}}$

C. $A=0$ and $B=\frac{1}{2}e^{\frac{13}{6}}$

D. $A=0$ and $B=2$

E. There are no such A and B that can make f continuous at $x=1$ and $x=2$.

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$

$\boxed{a^0 = 1}$

$\frac{19}{6} - 1 =$

$\frac{19}{6} - \frac{6}{6} = \frac{13}{6}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) \Rightarrow \frac{3+1+5}{A+2} = 5^0+2 \Rightarrow \frac{9}{A+2} = 3 \Rightarrow \boxed{A=1}$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) \Rightarrow \ln 2B + 1 = \frac{12+2+5}{4A+2} \Rightarrow \ln 2B + 1 = \frac{19}{6}$

$\Rightarrow \ln 2B = \frac{19}{6} - 1 \Rightarrow \ln 2B = \frac{13}{6}$

$\Rightarrow e^{\ln 2B} = e^{\frac{13}{6}} \Rightarrow 2B = e^{\frac{13}{6}}$

$\Rightarrow \boxed{B = \frac{1}{2} e^{\frac{13}{6}}}$

$\ln ab = \ln a + \ln b$
$\ln \frac{a}{b} = \ln a - \ln b$
$\ln 1 = 0$
$e^{\ln a} = a$
$\ln e^a = a$

6. (5 points) The solutions to the equation

$$\tan\left(\sin^{-1}(x)\right) = 3x$$

in the interval $(-1, 1)$ are

let $\sin^{-1}x = \theta \Rightarrow \sin\theta = x$

$\Rightarrow \tan\theta = 3x$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = 3x \Rightarrow \frac{x}{\cos\theta} = 3x$

$x \neq \pm\frac{2\sqrt{2}}{3}$ only

A. $x = \pm 1$

B. $x = 0$ and $x = \pm 1$

C. $x = 0$ **and** $x = \pm\frac{2\sqrt{2}}{3}$

D. $x = 0$

E. $x = 0$ and $x = \frac{1}{\sqrt{2}}$

أولاً $\frac{1}{x}$ $\boxed{x \neq 0}$ ؟

$\cancel{\frac{1}{x}} \cdot \frac{\cancel{x}}{\cos\theta} = \cancel{\frac{1}{x}} \cdot 3\cancel{x} \Rightarrow \frac{1}{\cos\theta} = 3 \Rightarrow \cos\theta = \frac{1}{3}$

ثانياً $\sin^2\theta + \cos^2\theta = 1 \Rightarrow x^2 + \frac{1}{9} = 1 \Rightarrow x^2 = 1 - \frac{1}{9} = \frac{9}{9} - \frac{1}{9} = \frac{8}{9}$

$\Rightarrow x^2 = \frac{8}{9} \Rightarrow x = \pm\sqrt{\frac{8}{9}} = \pm\frac{\sqrt{8}}{\sqrt{9}} = \pm\frac{\sqrt{2 \cdot 4}}{3} = \pm\frac{\sqrt{2}\sqrt{4}}{3} = \pm\frac{2\sqrt{2}}{3}$

نعوض 0 عشان نتأكد

$\tan\sin^{-1}x = 3x$

$x = 0 \Rightarrow \tan\underline{\sin^{-1}0} = 3 \cdot 0$ ؟

$\Rightarrow \tan 0 = 0 = 0$

يحقق المعادلة $x = 0$

$\Rightarrow x = 0$ or $x = \pm\frac{2\sqrt{2}}{3}$

7. (5 points) Suppose that $f(x)$ and $g(x)$ are differentiable functions defined for all x, and satisfy the following properties:

   - $f(-2) = g(-2)$
   - $f'(x) < g'(x)$ for all x.

   Which of the following statements are true?

   (i) $f(x) < g(x)$ for all $x > -2$.

   (ii) $f(x) < g(x)$ for all $x < -2$.

   (iii) The graphs of f and g do not intersect.

   (iv) The graphs of f and g intersect at exactly one point.

   (v) The graphs of f and g intersect at more than one point.

   A. (i) only B. (ii) and (iv) C. None of the above D. (ii) and (v) **E. (i) and (iv)**

Ex: $2x^2+3y^2=x+y$ find dy/dx. $4x+6y\frac{dy}{dx}=1+1.\frac{dy}{dx}$

$\Rightarrow 6y\frac{dy}{dx}-\frac{dy}{dx}=1-4x \Rightarrow \frac{dy}{dx}[6y-1]=1-4x \Rightarrow \frac{dy}{dx}=\frac{1-4x}{6y-1}$

نفس فكر الاشتقاق بس نجيب $\frac{dy}{dx}$

8. (10 points) Find $\frac{dy}{dx}$.

$f(x)=e^{x^2} \rightsquigarrow f'(x)=2x\,e^{x^2}$

$x^2+y^2=5 \Rightarrow 2x+2y\frac{dy}{dx}=0$

حول

a) $e^{\frac{x}{y}}=x-y$

$\Rightarrow e^{xy^{-1}}=x-y$ find $\frac{dy}{dx}$

power rule

$e^{xy^{-1}}[x(-1.y^{-2}\frac{dy}{dx})+y^{-1}.1]=1-1.\frac{dy}{dx}$

$\Rightarrow e^{x/y}\left[\frac{-x}{y^2}\frac{dy}{dx}+\frac{1}{y}\right]=1-\frac{dy}{dx}$

$\Rightarrow e^{x/y}\frac{-x}{y^2}\frac{dy}{dx}+\frac{e^{x/y}}{y}=1-\frac{dy}{dx}$

$\Rightarrow e^{\frac{x}{y}}\frac{-x}{y^2}\frac{dy}{dx}+\frac{dy}{dx}=1-\frac{e^{x/y}}{y}$

$\Rightarrow \frac{dy}{dx}\left[\frac{-x\,e^{x/y}}{y^2}+1\right]=1-\frac{e^{x/y}}{y}$

$$\Rightarrow \frac{dy}{dx}=\frac{1-\frac{e^{x/y}}{y}}{\frac{-x\,e^{x/y}}{y^2}+1}$$

$\frac{x}{y}=3$ find $\frac{dy}{dx}$.

$xy^{-1}=3$

ضرب

$x(-1y^{-2}\frac{dy}{dx})+y^{-1}.1=0$

$-\frac{x}{y^2}\frac{dy}{dx}+\frac{1}{y}=0$

$\Rightarrow -\frac{x}{y^2}\frac{dy}{dx}=-\frac{1}{y}$

$\frac{dy}{dx}=\frac{-1}{y}\cdot\frac{-y^2}{x}$

$\boxed{\frac{dy}{dx}=\frac{-y}{x}}$

9. (15 points)

a) (10 pts) Let $f(x) = \sin|x|$. Using the limit definition of the derivative, find the values of x for which $f(x)$ is differentiable, **and** find a formula for f'.

b) (5 pts) Let $f(x) = \dfrac{2^x + e^x}{3^x + 3^{-x}}$. Find $\left(f^{-1}\right)'(1)$.

10. (10 points) Determine whether the following statements are true or false. If true, justify your answer and if false, provide a counterexample.

a) The equation

$$\ln(x) - \frac{10}{x} = 0$$

has exactly one solution on the interval $\left[\frac{1}{2}, e^4\right]$.

True False

b) Let the function $f(x)$ be defined for all real numbers except $x = -4$. Furthermore, $f(x)$ satisfies the following inequality:

$$\left(\frac{4e^x - 11}{2e^x}\right)\tan^{-1}(x) \le f(x) \le \frac{\sqrt{(\pi x)^2 + 1}}{x + 4}.$$

Then, $\lim_{x\to\infty} f(x) = 2$.

True False